

Automorphism groups of simplicial complexes and rigidity for uniformly bounded representations

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Abstract

The aim of this paper is to extend the framework of W. Ballman and J. Świątkowski on L^2 -cohomology to reflexive Banach spaces. As an application, a spectral condition implying vanishing of cohomology for uniformly bounded representations on a Hilbert space is given.

1 Introduction

Since its introduction by David Kazhdan in [K], property (T) and its generalizations as cohomological vanishing has become a fundamental concept in mathematics [BHV]. The aim of this paper is to extend the framework of W. Ballman and J. Świątkowski [BS] to reflexive Banach spaces and as an application, to give a spectral condition implying vanishing of cohomology for uniformly bounded representations on a Hilbert space. Along with W. Ballman and J. Świątkowski, A. Żuk [Ż1] was among the first to also consider such criteria for unitary representations, both following fundamental work by H. Garland [G]. Since then, extending the spectral method beyond Hilbert spaces has been considered in [BFGM, CDH, DJ1, EJ-S, FM, L] and by Piotr W. Nowak [N] extending the spectral method in [Ż2] to reflexive Banach spaces. Appropriately extending the scheme in [N] we similarly extend the spectral condition of [BS] to uniformly bounded representations on a Hilbert space. Motivation for such generalizations arises, among others, from Shalom's conjecture [OWR] stating that any hyperbolic group Γ admits a uniformly bounded representation π with $\overline{H}^1(\Gamma, \pi) \neq 0$ together with a proper cocycle in $Z^1(\Gamma, \pi)$.

For a finite graph K with vertices \mathcal{V}_K , consider the graph Laplacian Δ_+ on the space of real valued functions on \mathcal{V}_K defined by

$$\Delta_+ f(v) = f(v) - Mf(v)$$

where $Mf(v)$ is the mean value of f on the vertices adjacent to v . Denote by $\lambda_1(K)$ the spectral gap of Δ_+ and its associated Poincaré constant by $\kappa_2(K, \mathbb{R}) = \lambda_1(K)^{-1/2}$. More generally $\kappa_2(K, \mathcal{H}) = \lambda_1(K)^{-1/2}$ for any separable infinite-dimensional Hilbert space \mathcal{H} , [N].

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Theorem. *Let X be a locally finite 2-dimensional simplicial complex, Γ a discrete properly discontinuous group of automorphisms of X and $\pi : \Gamma \rightarrow \mathcal{B}(\mathcal{H})$ a uniformly bounded representation of Γ on a separable infinite-dimensional Hilbert space \mathcal{H} . If for any vertex τ of X the link X_τ is connected and*

$$\sup_{g \in \Gamma} \|\pi_g\| < \frac{\sqrt{2}}{\kappa_2(X_\tau, \mathcal{H})},$$

then $L^2 H^1(X, \pi) = 0$.

Structure of the paper In Sections 2 to 7 the framework of [BS] for unitary representations on Hilbert spaces is extended to reflexive Banach spaces and isometric representations: Section 2 introduces the generalized set up; Section 3 and 4 deal with the dual of the twisted cochains; Section 5 introduces differentials and codifferentials; Section 6 discusses localization of the problem and Section 7 introduces the spectral set up in terms of Poincaré inequalities and constants on the links. Section 8 introduces L^p -cohomology as a natural extension to L^2 -cohomology, which is then applied to uniformly bounded representations using the fact they correspond to isometric representations on some reflexive Banach space.

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2 Set up

In this chapter notation is fixed. We recall the notation and some basic facts used by [BS] for weighted simplicial complexes and extend the notion of square integrable cochains to reflexive Banach spaces and $p > 1$.

2.1 Weighted complexes

Throughout, let X denote an n -dimensional locally finite simplicial complex. Following [BS] we use the following notation: $X(k)$ is the set of (unordered) k -simplexes of X ; $\Sigma(k)$ is the set of ordered k -simplexes of X . As usual we write $\sigma = \{v_0, \dots, v_k\}$ for a k -simplex and $\sigma = (v_0, \dots, v_k)$ for an ordered k -simplex. If the vertices of $\tau \in \Sigma(l)$ are vertices of $\sigma \in \Sigma(k)$, we say that $\tau \subset \sigma$, and for $\tau = (v_0, \dots, \hat{v}_i, \dots, v_k)$, i.e. $v_i \notin \tau$, we denote by $[\sigma : \tau] = (-1)^i$ the sign of τ in $\sigma = (v_0, \dots, v_k)$. As customary, we write σ_i for $(v_0, \dots, \hat{v}_i, \dots, v_k)$. In addition to orientation we consider X to be equipped with a weight ω , by which we mean a map from the oriented simplexes of X to the integers such that for $\sigma = (v_0, \dots, v_k) \in \Sigma(k)$,

$$\omega(\sigma) = \omega(\{v_0, \dots, v_k\}),$$

where $\omega(\{v_0, \dots, v_k\})$ denotes the number of n -simplexes containing $\{v_0, \dots, v_k\}$. In addition, we assume that $\omega(\sigma) \geq 1$ for every simplex of X . Beginning from Section 6 and onwards, we consider X locally through its links, where, by the link of $\tau = (v_0, \dots, v_l) \in \Sigma(l)$ denoted by X_τ , we mean the $(n-l-1)$ -dimensional subcomplex consisting of all simplexes $\{w_0, \dots, w_j\}$ disjoint from τ such that $\{v_0, \dots, v_l\} \cup \{w_0, \dots, w_j\}$ is a simplex of X . Since X is locally finite, X_τ is finite. Here as previously, $X_\tau(j)$ denotes the j -simplexes of X_τ , $\Sigma_\tau(j)$ its oriented j -simplexes and so on. In particular, for $\sigma \in X_\tau(j)$ and $\tau \in \Sigma(l)$ we denote by $\sigma * \tau \in \Sigma(j+l+1)$ the join of σ and τ obtained by juxtaposing the two in that order.

In addition to the above, we assume throughout that X is a Γ -space where Γ is a discrete topological group acting properly and discontinuously by simplicial automorphisms on X . In other words, Γ permutes the simplexes of X preserving their order and weights: that is for $\sigma = (v_0, \dots, v_k) \in \Sigma(k)$, $g \cdot \sigma = (g(v_0), \dots, g(v_k)) \in \Sigma(k)$ and $\omega(\sigma) = \omega(g \cdot \sigma)$. As usual, we denote by $\Gamma\sigma$ and Γ_σ the Γ -orbit and stabilizer of $\sigma \in \Sigma(k)$, respectively, by $\Sigma(k, \Gamma) \subset \Sigma(k)$ some chosen set of representatives of Γ -orbits in $\Sigma(k)$, and by $|\cdot|$ the counting measure on Γ . In particular since Γ is discrete, stabilizers are finite and the Haar measure on Γ is $|\cdot|$. Although the discreteness assumption can be avoided, it will be used when constructing projections in Section 3. For the following frequently used facts we refer to [BS]:

Proposition 1. [BS] *Let n be the dimension of X . Then, for $\tau \in \Sigma(k)$*

$$\sum_{\substack{\sigma \in \Sigma(k+1) \\ \tau \subset \sigma}} \omega(\sigma) = (n-k)(k+2)! \omega(\tau).$$

□

Proposition 2. [BS] *For $0 \leq l < k \leq n$, let $f = f(\tau, \sigma)$ be a Γ -invariant function on the set of pairs (τ, σ) , $\tau \in \Sigma(l)$, $\sigma \in \Sigma(k)$, such that $\tau \subset \sigma$. Then*

$$\sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\tau \in \Sigma(l) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\sigma|} = \sum_{\tau \in \Sigma(l, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\tau|},$$

whenever either side is absolutely convergent.

□

More generally, Proposition 2 holds for locally compact unimodular groups [DJ2] replacing the counting measure with the Haar measure.

2.2 Banach space setting, isometric representations and p -integrable cochains

Throughout, let $(E, \|\cdot\|_E)$ denote a reflexive Banach space, $\langle \cdot, \cdot \rangle_E$ the natural pairing between E and its continuous dual E^* , \simeq isomorphism, \cong isometric isomorphism and p^* the adjoint index of p such that $1/p + 1/p^* = 1$. Moreover, let $\pi : \Gamma \rightarrow \text{Iso}(E)$ denote an isometric representation of Γ on E where $\text{Iso}(E)$ denotes the group of isometric linear automorphisms on E and by $\bar{\pi} : \Gamma \rightarrow \text{Iso}(E^*)$ its corresponding contragradient representation given by $\bar{\pi}_g = \pi_{g^{-1}}^*$ where π^* is the transpose of π . For combinatorial purposes we also introduce antisymmetrization:

Definition 3. For $n \geq 1$ we denote by S_n the symmetric group of n elements and by $\text{sign}: S_n \rightarrow \{-1, 1\}$ the signature of the permutation: 1 if $\alpha \in S_n$ is an even permutation of the n elements and otherwise -1 . For $f: \Sigma(k) \rightarrow E$ define its alternation point-wise as the linear idempotent map

$$\text{Alt}f(\sigma) = \frac{1}{(k+1)!} \sum_{\alpha \in S_{k+1}} \text{sign}(\alpha) \alpha^* f(\sigma),$$

where $\alpha^* f(\sigma) = f(v_{\alpha(0)}, \dots, v_{\alpha(k)})$ for $\sigma = (v_0, \dots, v_k) \in \Sigma(k)$. As usual, we say that f is alternating if $\text{Alt}f = f$, and symmetric if $\text{Alt}f = 0$.

Replacing inner product with dual pairing and unitary representations by isometric representations, we next introduce twising and cochains as in [BS].

Definition 4. Let $\mathcal{E}^{(k,p)}(X, E)$ denote the semi-normed vector space of k -cochains $f: \Sigma(k) \rightarrow E$ for which the semi norm given by

$$\|f\|_{(k,p)} = \left(\sum_{\sigma \in \Sigma(k, \Gamma)} \|f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \right)^{1/p},$$

is finite.

Definition 5. For $f \in \mathcal{E}^{(k,p)}(X, E)^*$, we denote by

$$\langle \phi, f \rangle_k = \sum_{\sigma \in \Sigma(k, \Gamma)} \langle \phi(\sigma), f(\sigma) \rangle_E \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|}$$

the dual pairing between $\mathcal{E}^{(k,p)}(X, E)$ and $\mathcal{E}^{(k,p)}(X, E)^*$.

Proposition 6. $\mathcal{E}^{(k,p)}(X, E)^* \cong \mathcal{E}^{(k,p^*)}(X, E^*)$. □

Definition 7. Let $f: \Sigma(k) \rightarrow E$. If for every $g \in \Gamma$ and every $\sigma \in \Sigma(k)$

$$f(g \cdot \sigma) = \pi_g \cdot f(\sigma),$$

then we say that f is twisted by π , or for short just twisted.

Definition 8. Let $C^{(k,p)}(X, E)$ denote the vector space of all alternating maps $f: \Sigma(k) \rightarrow E$ twisted by π .

Those alternating maps twisted by π whose $\|\cdot\|_{(k,p)}$ norm is finite are called p -integrable mod Γ and we use the following notation:

Definition 9. Let $L^{(k,p)}(X, E) = \{f \in C^{(k,p)}(X, E): \|f\|_{(k,p)} < \infty\}$ denote the vector subspace of all alternating k -cochains of X twisted by π .

In particular, if Γ acts cocompactly on X , then $L^{(k,p)}(X, E) = C^{(k,p)}(X, E)$ since then X/Γ is compact, the set of representatives $\Sigma(k, \Gamma)$ is finite, and $\|f\|_{(k,p)} < \infty$ for all $f \in C^{(k,p)}(X, E)$.

We end this section by proving that $L^{(k,p)}(X, E)$ is a normed space with respect to $\|\cdot\|_{(k,p)}$. Towards this end we first show that $\|\cdot\|_{(k,p)}$ is independent of the set of representatives when $f \in L^{(k,p)}(X, E)$.

Lemma 10. *If $f \in L^{(k,p)}(X, E)$, then $\|f\|_{(k,p)}$ is independent of the choice of $\Sigma(k, \Gamma)$.*

Proof. Let $\Sigma'(k, \Gamma)$ be another set of representatives. Then,

$$\begin{aligned} \|f\|_{(k,p)}^p &= \sum_{\sigma' \in \Sigma'(k, \Gamma)} \|f(\sigma')\|_E^p \frac{\omega(\sigma')}{(k+1)!|\Gamma_{\sigma'}|} = \sum_{\sigma' \in \Sigma'(k, \Gamma)} \|f(g' \cdot \sigma')\|_E^p \frac{\omega(g' \cdot \sigma')}{(k+1)!|\Gamma_{g' \cdot \sigma'}|} \\ &= \sum_{\sigma \in \Sigma(k, \Gamma)} \|f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)!|\Gamma_{\sigma}|}, \end{aligned}$$

choosing for each $\sigma' \in \Sigma'(k, \Gamma)$ a $g' \in \Gamma$ such that $g' \cdot \sigma' = \sigma \in \Sigma(k, \Gamma)$ and observing that f is twisted by π and both the norm and ω are Γ -invariant. \square

Proposition 11. *$L^{(k,p)}(X, E) \subseteq \mathcal{E}^{(k,p)}(X, E)$ is a normed vector space.*

Proof. It suffices to show that the seminorm $\|\cdot\|_{(k,p)}$ on $\mathcal{E}^{(k,p)}(X, E)$ restricted to $L^{(k,p)}(X, E)$ is a norm. To this end, suppose $\|f\|_{(k,p)} = 0$ for $f \in L^{(k,p)}(X, E)$. By Lemma 10 we may assume $f(\sigma) = 0$ for all $\sigma \in \Sigma(k, \Gamma)$. Since $f(g \cdot \sigma) = \pi_g f(\sigma)$ and the action of Γ is transitive on the orbits it follows that $f(\sigma) = 0$ for all $\sigma \in \Sigma(k)$. \square

3 Projecting k -cochains onto $L^{(k,p)}(X, E)$

In order to extend the framework of [BS], the dual space of the alternating and twisted cochains has at first to be identified up to isometric isomorphism. Following the scheme presented in [N], we begin by stepwise constructing a continuous projection P_L from $\mathcal{E}^{(k,p)}(X, E)$ onto $L^{(k,p)}(X, E)$.

Definition 12. *Define $\tilde{P} : \mathcal{E}^{(k,p)}(X, E) \rightarrow \mathcal{E}^{(k,p)}(X, E)$ by*

$$\tilde{P}f(\sigma) = \begin{cases} \sum_{s \in \Gamma_\sigma} \pi_s f'(\sigma) & \text{if } \sigma \in \Sigma(k, \Gamma) \\ \sum_{\substack{h \in \Gamma \\ h \cdot \tau = \sigma}} \pi_h f'(\tau) & \text{if } \sigma \notin \Sigma(k, \Gamma) \text{ for } \tau \in \Sigma(k, \Gamma), \end{cases}$$

where $f' : \Sigma(k, \Gamma) \rightarrow E$ is the restriction of $f : \Sigma(k) \rightarrow E$ to $\Sigma(k, \Gamma)$.

The this map is well defined, in particular we note that $\{h \in \Gamma : h \cdot \tau = \sigma\} = h\Gamma_\tau$. As the following proposition shows, \tilde{P} maps k -cochains to k -cochains twisted by π .

Proposition 13. *For $f \in \mathcal{E}^{(k,p)}(X, E)$, the k -cochain $\tilde{P}f : \Sigma(k) \rightarrow E$ is twisted by π .*

Proof. Let $\sigma \in \Sigma(k)$. Then either $\sigma \in \Sigma(k, \Gamma)$ or $\sigma \notin \Sigma(k, \Gamma)$. Suppose at first $\sigma \in \Sigma(k, \Gamma)$. If $g \in \Gamma_\sigma$, then clearly $\pi_g \tilde{P}f(\sigma) = \tilde{P}f(g \cdot \sigma)$. On the other hand, if $g \notin \Gamma_\sigma$ we get

$$\tilde{P}f(g \cdot \sigma) = \sum_{\substack{h \in \Gamma \\ h \cdot \sigma = g \cdot \sigma}} \pi_h f'(\sigma),$$

and

$$\pi_g \tilde{P}f(\sigma) = \sum_{h \in g\Gamma_\sigma} \pi_h f'(\sigma).$$

But $\{h \in \Gamma : h \cdot \sigma = g \cdot \sigma\} = \{h \in \Gamma : h \in g\Gamma_\sigma\}$, so the claim holds for $\sigma \in \Sigma(k, \Gamma)$. Suppose $\sigma \notin \Sigma(k, \Gamma)$. If $g \cdot \sigma \in \Sigma(k, \Gamma)$, then

$$\tilde{P}f(g \cdot \sigma) = \sum_{s \in \Gamma_{g \cdot \sigma}} \pi_s f'(g \cdot \sigma) = \sum_{s \in g\Gamma_\sigma g^{-1}} \pi_s f'(g \cdot \sigma) = \sum_{h \in \Gamma_\sigma} \pi_{ghg^{-1}} f'(g \cdot \sigma),$$

as $\Gamma_{g \cdot \sigma} = g\Gamma_\sigma g^{-1}$, and so

$$\begin{aligned} \pi_g \tilde{P}f(\sigma) &= \sum_{\substack{h \in \Gamma \\ hg \cdot \sigma = \sigma}} \pi_{gh} f'(g \cdot \sigma) = \sum_{\substack{h \in \Gamma \\ hg \in \Gamma_\sigma}} \pi_{gh} f'(g \cdot \sigma) = \sum_{h \in \Gamma_\sigma g^{-1}} \pi_{gh} f'(g \cdot \sigma) \\ &= \sum_{h \in \Gamma_\sigma} \pi_{ghg^{-1}} f'(g \cdot \sigma) = \tilde{P}f(g \cdot \sigma). \end{aligned}$$

On the other hand, if $g \cdot \sigma \notin \Sigma(k, \Gamma)$, write $\tilde{P}f(g \cdot \sigma) = \sum_{h \in A} \pi_h f'(\tau)$ where $A = \{h \in \Gamma : h \cdot \tau = g \cdot \sigma\} = gB$ for $B = \{s \in \Gamma : s \cdot \tau = \sigma\}$ and $\tau \in \Sigma(k, \Gamma)$. Hence,

$$\begin{aligned} \pi_g \tilde{P}f(\sigma) &= \sum_{\substack{s \in \Gamma \\ s \cdot \tau = \sigma}} \pi_{gs} f'(\tau) = \sum_{h \in g\{s \in \Gamma : s \cdot \tau = \sigma\}} \pi_h f'(\tau) = \sum_{h \in gB} \pi_h f'(\tau) \\ &= \sum_{h \in A} \pi_h f'(\tau) = \tilde{P}f(g \cdot \sigma), \end{aligned}$$

so the claim holds for $\sigma \notin \Sigma(k, \Gamma)$ as well. \square

Recalling that Γ is discrete, normalizing \tilde{P} as below gives a projection onto the twisted cochains.

Definition 14. Define $P : \mathcal{E}^{(k,p)}(X, E) \rightarrow \mathcal{E}^{(k,p)}(X, E)$ by

$$Pf(\sigma) = \frac{1}{|\Gamma_\sigma|} \tilde{P}f(\sigma).$$

Proposition 15. P is a projection onto the twisted cochains.

Proof. Clearly $P^2 = P$ and onto. Now, suppose f is twisted and $\sigma \in \Sigma(k)$. If $\sigma \in \Sigma(k, \Gamma)$, then, recalling the discreteness assumption

$$Pf(\sigma) = \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \pi_s f'(\sigma) = \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} f'(s \cdot \sigma) = \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} f'(\sigma) = f'(\sigma) = f(\sigma).$$

Similarly, for $\sigma \notin \Sigma(k, \Gamma)$

$$Pf(\sigma) = \frac{1}{|\Gamma_\sigma|} \sum_{\substack{h \in \Gamma \\ h \cdot \tau = \sigma}} \pi_h f'(\tau) = \frac{1}{|\Gamma_\sigma|} \sum_{\substack{h \in \Gamma \\ h \cdot \tau = \sigma}} f(h \cdot \tau) = \frac{1}{|\Gamma_\sigma|} \sum_{\substack{h \in \Gamma \\ h \cdot \tau = \sigma}} f(\sigma) = f(\sigma),$$

as $|\{h \in \Gamma : h \cdot \tau = \sigma\}| = |h\Gamma_\tau| = |\Gamma_\tau|$ and $|\Gamma_\sigma| = |h\Gamma_\tau h^{-1}| = |\Gamma_\tau|$. \square

Corollary 16. *P is continuous with $\|Pf\|_{(k,p)} \leq \|f\|_{(k,p)}$ for $f \in \mathcal{E}^{(k,p)}(X, E)$ with equality for $f \in L^{(k,p)}(X, E)$.*

Proof. A straightforward consequence of Definition 14, and the observation that $Pf = f$ for $f \in L^{(k,p)}(X, E)$. \square

Thus, we have constructed a projection P onto the cochains twisted by π . However, cochains in the image are not necessarily alternating and hence not necessarily in $L^{(k,p)}(X, E)$. Antisymmetrizing P fixes this. We begin with the following observation:

Corollary 17. *If f is twisted, then $\text{Alt} f$ is twisted.*

Proof. Suppose $f : \Sigma(k) \rightarrow E$ is twisted. Then, $\text{Alt} f(g \cdot \sigma) = \text{Alt}(\pi_g f(\sigma)) = \pi_g(\text{Alt} f(\sigma))$ for all $g \in \Gamma$ and $\sigma \in \Sigma(k)$, where we used the fact that f is twisted in the first equality and linearity of π_g in the last equality. Hence, $\text{Alt} f$ is twisted as well. \square

Corollary 18. *Suppose $f \in \mathcal{E}^{(k,p)}(X, E)$, then*

$$\|\text{Alt} f\|_{(k,p)}^p \leq (k+1)! \|f\|_{(k,p)}^p$$

Proof. Since

$$\begin{aligned} \|\text{Alt} f(\sigma)\|_E^p &= \frac{1}{(k+1)!^p} \left\| \sum_{\alpha \in S_{k+1}} \text{sign}(\alpha) \alpha^* f(\sigma) \right\|_E^p \leq \frac{(k+1)!^p}{(k+1)!^p} \sum_{\alpha \in S_{k+1}} \|\alpha^* f(\sigma)\|_E^p \\ &= \sum_{\alpha \in S_{k+1}} \|\alpha^* f(\sigma)\|_E^p, \end{aligned}$$

it follows that

$$\begin{aligned} \|\text{Alt} f\|_{(k,p)}^p &= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\text{Alt} f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\ &\leq \sum_{\alpha \in S_{k+1}} \sum_{\sigma \in \Sigma(k, \Gamma)} \|\alpha^* f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\ &= (k+1)! \sum_{\sigma \in \Sigma(k, \Gamma)} \|f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \end{aligned}$$

since we sum over all representatives in the last equality, and for $\sigma = (v_0, \dots, v_k)$, $\omega(v_{\alpha(0)}, \dots, v_{\alpha(k)}) = \omega(\sigma)$ and $\Gamma_{(v_{\alpha(0)}, \dots, v_{\alpha(k)})} = \Gamma_\sigma$ for all $\alpha \in S_{k+1}$. \square

Proposition 19. *The map $P_L : \mathcal{E}^{(k,p)}(X, E) \rightarrow \mathcal{E}^{(k,p)}(X, E)$, given by*

$$P_L = \text{Alt} \circ P$$

defines a projection onto $L^{(k,p)}(X, E)$. In other words, the diagram

$$\begin{array}{ccc} \mathcal{E}^{(k,p)}(X, E) & \xrightarrow{P} & \{\text{k-cochain}_\pi\} \\ & \searrow P_L & \downarrow \text{Alt} \\ & & L^{(k,p)}(X, E) \end{array}$$

commutes.

Proof. Clearly $P_L^2 = P$. By Proposition 15 P is a projection onto the twisted cochains, and since Alt preserves twisting by Corollary 17, P_L is a projection onto $L^{(k,p)}(X, E)$. \square

Proposition 20. P_L is continuous with $\|P_L f\|_{(k,p)}^p \leq (k+1)! \|f\|_{(k,p)}^p$.

Proof.

$$\begin{aligned}
\|P_L f\|_{(k,p)}^p &= \sum_{\sigma \in \Sigma(k, \Gamma)} \|P_L f(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \left\| \text{Alt} \left(\frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \pi_s f'(\sigma) \right) \right\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \left\| \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \pi_s \text{Alt} f'(\sigma) \right\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&\leq \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{1}{|\Gamma_\sigma|^p} |\Gamma_\sigma|^p \max_{s \in \Gamma_s} \{ \|\pi_s \text{Alt} f'(\sigma)\|_E^p \} \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\text{Alt} f'(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&\leq \sum_{\sigma \in \Sigma(k, \Gamma)} (k+1)! \|f'(\sigma)\|_E^p \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \\
&= (k+1)! \|f\|_{(k,p)}^p,
\end{aligned}$$

where we have used Corollary 18 in the last inequality. \square

Corollary 21. $L^{(k,p)}(X, E) \subseteq \mathcal{E}^{(k,p)}(X, E)$ is closed. \square

4 $L^{(k,p)}(X, E)^* \cong L^{(k,p^*)}(X, E^*)$

Having constructed a continuous projection from $\mathcal{E}^{(k,p)}(X, E)$ onto $L^{(k,p)}(X, E)$ we show that the dual of $L^{(k,p)}(X, E)$ can be identified up to isometric isomorphism with $\mathcal{E}^{(k,p^*)}(X, E^*)/\text{Ann}(L^{(k,p)}(X, E))$, cf. Corollary 24, and finally that the latter is isometrically isomorphic to $L^{(k,p^*)}(X, E^*)$, cf. Proposition 26, 26 and 28 below. Towards this end, recall that by the annihilator of a subspace $M \subseteq E$ we mean the vector space $\text{Ann}(M) = \{x \in E^* : \langle y, x \rangle_E = 0 \ \forall y \in M\}$ of all bounded linear functionals on E that vanish on M . The following fact contains the idea of the proof:

Proposition 22. [D] Suppose E is a Banach space such that $E = M \oplus N$ and denote by P the corresponding projection onto M . Then,

1. $\ker P^* = \text{Ann}(M)$ and $\text{im } P^* = \text{Ann}(N)$;
2. $E^* \simeq \text{Ann}(N) \oplus \text{Ann}(M)$;
3. if M is closed $M^* \cong E^*/\text{Ann}(M)$. \square

Let $L_-^{(k,p)}(X, E)$ denote the closed complement of $L^{(k,p)}(X, E)$ in $\mathcal{E}^{(k,p)}(X, E)$. That is $L_-^{(k,p)}(X, E) = \ker P_L$, or in other words:

Corollary 23. $L_-^{(k,p)}(X, E) = \{f \in \mathcal{E}^{(k,p)}(X, E) : \text{Alt} f(\sigma) = 0 \ \forall \sigma \in \Sigma(k, \Gamma)\}$ is a closed subspace of $\mathcal{E}^{(k,p)}(X, E)$

Proof. Given $f \in L_-^{(k,p)}(X, E)$, $(I - P_L)f(\sigma) = f(\sigma)$ for all $\sigma \in \Sigma(k)$, and hence for all $\sigma \in \Sigma(k, \Gamma)$

$$(I - P_L)f'(\sigma) = f'(\sigma) - P_L f'(\sigma) = f'(\sigma) - \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \pi_s \text{Alt} f'(\sigma) = f'(\sigma),$$

implying by linearity that $\text{Alt} f'(\sigma) = 0$. Hence, f is symmetric on representatives. \square

Corollary 24. $L^{(k,p)}(X, E)^* \cong \mathcal{E}^{(k,p^*)}(X, E^*) / \text{Ann}(L^{(k,p)}(X, E))$.

Proof. Since $L^{(k,p)}(X, E)$ is a closed subspace of $\mathcal{E}^{(k,p)}(X, E)$ by Corollary 21, the claim now follows from Proposition 22(3) and the fact that $\mathcal{E}^{(k,p)}(X, E)^* \cong \mathcal{E}^{(k,p^*)}(X, E^*)$. \square

It now remains to identify the annihilators, cf. Proposition 26, to prove isomorphism and finally isometry. As indicated by Proposition 22 this requires knowing P_L^* .

Proposition 25. Let $\overline{P}_L : \mathcal{E}^{(k,p^*)}(X, E^*) \rightarrow \mathcal{E}^{(k,p^*)}(X, E^*)$ be a projection as above. Then $\overline{P}_L = P_L^*$.

Proof. Assume first $k = 1$, let $f \in \mathcal{E}^{(1,p)}(X, E)$ and $\phi \in \mathcal{E}^{(1,p^*)}(X, E^*)$. For $\sigma = (v_0, v_1) \in \Sigma(1, \Gamma)$ we denote by $-\sigma$ the simplex (v_1, v_0) . Now,

$$\begin{aligned} \langle P_L f, \phi \rangle_1 &= \sum_{\sigma \in \Sigma(1, \Gamma)} \langle P_L f(\sigma), \phi(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &\stackrel{(*)}{=} \sum_{\sigma \in \Sigma(1, \Gamma)} \left\langle \frac{1}{|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \pi_s \left(\frac{1}{2} f'(\sigma) - \frac{1}{2} f'(-\sigma) \right), \phi'(\sigma) \right\rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &= \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle \pi_s (f'(\sigma) - f'(-\sigma)), \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &= \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle \pi_s f'(\sigma), \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &\quad - \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle \pi_s f'(-\sigma), \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &\stackrel{(**)}{=} \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle \pi_s f'(\sigma), \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\ &\quad - \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle \pi_s f'(\sigma), \phi'(-\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle f'(\sigma), \bar{\pi}_s \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\
&\quad - \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle f'(\sigma), \bar{\pi}_s \phi'(-\sigma) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(1, \Gamma)} \frac{1}{2|\Gamma_\sigma|} \sum_{s \in \Gamma_\sigma} \langle f'(\sigma), \bar{\pi}_s (\phi'(\sigma) - \phi'(-\sigma)) \rangle_E \frac{\omega(\sigma)}{2!|\Gamma_\sigma|} \\
&= \langle f, \bar{P}_L \phi \rangle_1,
\end{aligned}$$

where $(*)$ and the last equality follow from the definition of P_L and \bar{P}_L , respectively when $k = 1$. $(**)$ follows as we sum over all $\sigma \in \Sigma(1, \Gamma)$, so the sums where we switch the summation variable σ with $-\sigma$ agree as $\omega(\sigma) = \omega(-\sigma)$. For $k > 1$ the calculation goes similarly, denoting $\sigma = (v_0, \dots, v_k) \in \Sigma(k, \Gamma)$ and arguing similarly,

$$\begin{aligned}
\langle P_L f, \phi \rangle_k &= \sum_{\sigma \in \Sigma(k, \Gamma)} \langle P_L f(\sigma), \phi(\sigma) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{s \in \Gamma_s} \sum_{\alpha \in S_{k+1}} \frac{1}{|\Gamma_\sigma|} \frac{1}{(k+1)!} (-1)^{\text{sgn}(\alpha)} \langle \pi_s f'((v_{\alpha(0)}, \dots, v_{\alpha(k)})), \phi'(\sigma) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{s \in \Gamma_s} \sum_{\alpha \in S_{k+1}} \frac{1}{|\Gamma_\sigma|} \frac{1}{(k+1)!} (-1)^{\text{sgn}(\alpha)} \langle \pi_s f'(\sigma), \phi'((v_{\alpha(0)}, \dots, v_{\alpha(k)})) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{s \in \Gamma_s} \sum_{\alpha \in S_{k+1}} \frac{1}{|\Gamma_\sigma|} \frac{1}{(k+1)!} (-1)^{\text{sgn}(\alpha)} \langle f'(\sigma), \bar{\pi}_s \phi'((v_{\alpha(0)}, \dots, v_{\alpha(k)})) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \langle f, \bar{P}_L \phi \rangle_k,
\end{aligned}$$

where the first and last equality follows by the definition of P_L and linearity of the dual pairing, and the third similarly as in the case $k = 1$ above. \square

Proposition 26. *The following are equal as sets:*

1. $\text{Ann}(L_-^{(k,p)}(X, E)) = L^{(k,p^*)}(X, E^*);$
2. $\text{Ann}(L^{(k,p)}(X, E)) = L_-^{(k,p^*)}(X, E^*).$

Proof. Suppose $f \in L_-^{(k,p)}(X, E)$ and $\phi \in \mathcal{E}^{(k,p^*)}(X, E^*)$. Then,

$$\begin{aligned}
\langle f, \phi \rangle_k &= \sum_{\sigma \in \Sigma(k, \Gamma)} \langle f(\sigma), \phi(\sigma) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \langle (I - P_L)f(\sigma), \phi(\sigma) \rangle_E \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \langle f, \phi \rangle_k - \langle P_L f, \phi \rangle_k = \langle f, \phi \rangle_k - \langle f, \bar{P}_L \phi \rangle_k.
\end{aligned}$$

Hence, $\langle f, \bar{P}_L \phi \rangle_E = 0$ for all $f \in L_-^{(k,p)}(X, E)$ where $\bar{P}_L \phi \in L^{(k,p^*)}(X, E^*)$. Thus, $L^{(k,p^*)}(X, E^*) \subseteq \text{Ann}(L_-^{(k,p)}(X, E))$. On the other hand, suppose $\phi \in \text{Ann}(L_-^{(k,p)}(X, E))$, then $\langle f, \phi \rangle_k = 0$ for all $f \in L_-^{(k,p)}(X, E)$. Hence, for all

$f \in \mathcal{E}^{(k,p^*)}(X, E^*)$, so $0 = \langle (I - P_L)f, \phi \rangle_k$ if and only if $\langle f, \phi \rangle_k = \langle P_L f, \phi \rangle_k = \langle f, \overline{P}_L \phi \rangle_k$ for all $f \in \mathcal{E}^{(k,p^*)}(X, E^*)$. Thus, $\phi = \overline{P}_L \phi$ implies that $\phi \in L^{(k,p^*)}(X, E^*)$, so $\text{Ann}(L_-^{(k,p)}(X, E)) \subseteq L^{(k,p^*)}(X, E^*)$ proving the first claim. The proof of the second claim goes similarly. \square

Corollary 27. *The following are isomorphic:*

1. $L^{(k,p)}(X, E)^* \cong \mathcal{E}^{(k,p^*)}(X, E^*) / L_-^{(k,p^*)}(X, E^*) \simeq L^{(k,p^*)}(X, E^*)$;
2. $L_-^{(k,p)}(X, E)^* \cong \mathcal{E}^{(k,p^*)}(X, E^*) / L^{(k,p^*)}(X, E^*) \simeq L_-^{(k,p^*)}(X, E^*)$.

Proof. The first isomorphism follows immediately combining Propositions 22 and 26, and the latter isomorphism by Proposition 22. \square

Proposition 28. *The following are isometrically isomorphic:*

1. $L^{(k,p)}(X, E)^* \cong L^{(k,p^*)}(X, E^*)$;
2. $L_-^{(k,p)}(X, E)^* \cong L_-^{(k,p^*)}(X, E^*)$.

Proof. Consider the second claim. Consider $\mathcal{E}^{(k,p^*)}(X, E^*) / L^{(k,p^*)}(X, E^*)$ consisting of the cosets $[\phi] = \phi + L^{(k,p^*)}(X, E^*)$ for $\phi \in \mathcal{E}^{(k,p^*)}(X, E^*)$. We claim that if $\phi \in L_-^{(k,p^*)}(X, E^*)$, then $\|[\phi]\| = \|\phi\|_{(k,p^*)}$ where $\|\cdot\| = \inf\{\|\xi\|_{(k,p^*)} : \xi \in [\phi]\}$ is the quotient norm. On the other hand, $\mathcal{E}^{(k,p^*)}(X, E^*) / L^{(k,p^*)}(X, E^*) \simeq L_-^{(k,p^*)}(X, E^*)$ by Corollary 27, so $\mathcal{E}^{(k,p^*)}(X, E^*) / L^{(k,p^*)}(X, E^*) \cong L_-^{(k,p^*)}(X, E^*)$ and consequently $L_-^{(k,p)}(X, E)^* \cong L_-^{(k,p^*)}(X, E^*)$. Towards this end, fix $\phi \in L_-^{(k,p^*)}(X, E^*)$. Thus, for $\psi \in L^{(k,p^*)}(X, E^*)$ we have

$$\begin{aligned}
\|\phi + \psi\|_{(k,p^*)}^{p^*} &= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\phi(\sigma) + \psi(\sigma)\|_{E^*}^{p^*} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\phi(\sigma) - \psi(-\sigma)\|_{E^*}^{p^*} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\phi(-\sigma) - \psi(-\sigma)\|_{E^*}^{p^*} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \|\phi(-\sigma) - \psi(-\sigma)\|_{E^*}^{p^*} \frac{\omega(-\sigma)}{(k+1)!|\Gamma_{-\sigma}|} \\
&= \|\phi - \psi\|_{(k,p^*)}^{p^*},
\end{aligned}$$

where $-\sigma = (v_1, v_0, v_2, \dots, v_k)$ for $\sigma = (v_0, \dots, v_k)$. The second equality follows since ψ is alternating, the third equality since ϕ is symmetric on representatives, the fourth since ω is symmetric and $\Gamma_\sigma = \Gamma_{-\sigma}$, and the last equality holds since we sum over all $\sigma \in \Sigma(k, \Gamma)$, which contains all the oriented simplexes with the vertices of σ . Thus, $\|\phi + \psi\|_{(k,p^*)}^{p^*} = \|\phi - \psi\|_{(k,p^*)}^{p^*}$ and consequently by the triangle inequality

$$2\|\phi\|_{(k,p^*)} = \|2\phi + \psi - \psi\|_{(k,p^*)} \leq \|\phi + \psi\|_{(k,p^*)} + \|\phi - \psi\|_{(k,p^*)},$$

and so

$$\|\phi\|_{(k,p^*)} \leq \frac{1}{2}(\|\phi + \psi\|_{(k,p^*)} + \|\phi - \psi\|_{(k,p^*)}) = \|\phi + \psi\|_{(k,p)}.$$

Now, taking the infimum over $\psi \in L^{(k,p^*)}(X, E^*)$ thus shows that the quotient norm of $[\phi]$ is $\|\phi\|_{(k,p^*)}$, proving the second claim. The first claim is proven similarly by considering the cosets in $\mathcal{E}^{(k,p^*)}(X, E^*) / L_-^{(k,p^*)}(X, E^*)$.

□

5 Differentials and codifferentials

Having identified the dual of $L^{(k,p)}(X, E)$ up to isometric isomorphism we extend the notion of differentials and codifferentials as presented in [BS] to reflexive Banach spaces.

Definition 29. *Codifferentials and differentials. Define the codifferential*

$$d_k : \mathcal{E}^{(k,p)}(X, E) \rightarrow \mathcal{E}^{(k+1,p)}(X, E)$$

point-wise by

$$d\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i),$$

and the differential

$$\delta_{k+1} : L^{(k+1,p^*)}(X, E^*) \rightarrow L^{(k,p^*)}(X, E^*),$$

as the adjoint of d given by $\langle \phi, d\psi \rangle_{k+1} = \langle \delta\phi, \psi \rangle_k$ for all $\psi \in L^{(k,p)}(X, E)$ and $\phi \in L^{(k+1,p^)}(X, E^*)$.*

Similarly, we denote by $\bar{d}_k : \mathcal{E}^{(k,p^*)}(X, E^*) \rightarrow \mathcal{E}^{(k+1,p^*)}(X, E^*)$ the map given by $\bar{d}\psi(\sigma) = \sum_{i=0}^k (-1)^i \psi(\sigma_i)$ for $\psi \in \mathcal{E}^{(k,p^*)}(X, E^*)$, and likewise for the differential.

Corollary 30.

$$\dots \xrightarrow{\delta_{k+2}} L^{(k+1,p^*)}(X, E^*) \xrightarrow{\delta_{k+1}} L^{(k,p^*)}(X, E^*) \xrightarrow{\delta_k} \dots$$

is a chain complex over \mathbb{R} dual to the cochain complex

$$\dots \xleftarrow{d_{k+1}} L^{(k+1,p)}(X, E) \xleftarrow{d_k} L^{(k,p)}(X, E) \xleftarrow{d_{k-1}} \dots$$

□

As the following shows, both d and δ are bounded operators.

Proposition 31. *Let $\phi \in L^{(k,p)}(X, E)$. Then $d : L^{(k,p)}(X, E) \rightarrow L^{(k+1,p)}(X, E)$ is a bounded operator and*

$$\|d\phi\|_{(k+1,p)}^p \leq (n-k)(k+2)^p \|\phi\|_{(k,p)}^p.$$

Proof. Using the estimate

$$\begin{aligned}
\left\| \sum_{i=0}^{k+1} (-1)^i \phi(s_i) \right\|_E^p &\leq \left(\|\phi(s_0)\|_E + \left\| \sum_{i=1}^{k+1} (-1)^i \phi(s_i) \right\|_E \right)^p \leq \left(\sum_{i=0}^{k+1} \|\phi(s_i)\|_E \right)^p \\
&\leq ((k+2) \max\{\|\phi(s_0)\|, \dots, \|\phi(s_{k+1})\|\})^p \\
&\leq (k+2)^p \sum_{i=0}^{k+1} \|\phi(s_i)\|_E^p,
\end{aligned}$$

it follows that

$$\begin{aligned}
\|d\phi\|_{(k+1,p)}^p &= \sum_{s \in \Sigma(k+1, \Gamma)} \|d\phi(s)\|_E^p \frac{\omega(s)}{(k+2)!|\Gamma_s|} \\
&= \sum_{s \in \Sigma(k+1, \Gamma)} \left\| \sum_{i=0}^{k+1} (-1)^i \phi(s_i) \right\|_E^p \frac{\omega(s)}{(k+2)!|\Gamma_s|} \\
&\leq \sum_{s \in \Sigma(k+1, \Gamma)} \left(\frac{(k+2)^p \omega(s)}{(k+2)!|\Gamma_s|} \sum_{i=0}^{k+1} \|\phi(s_i)\|_E^p \right) \\
&= \sum_{s \in \Sigma(k+1, \Gamma)} \left(\frac{(k+2)^p \omega(s)}{(k+2)!(k+1)!|\Gamma_s|} \sum_{\substack{t \in \Sigma(k) \\ t \subset s}} \|[s:t]\phi(t)\|_E^p \right) \\
&= \sum_{s \in \Sigma(k+1, \Gamma)} \left(\frac{(k+2)^p \omega(s)}{(k+2)!(k+1)!|\Gamma_s|} \sum_{\substack{t \in \Sigma(k) \\ t \subset s}} \|\phi(t)\|_E^p \right) \\
&\stackrel{(*)}{=} \sum_{t \in \Sigma(k, \Gamma)} \left(\frac{(k+2)^p \|\phi(t)\|_E^p}{(k+2)!(k+1)!|\Gamma_t|} \sum_{\substack{s \in \Sigma(k+1) \\ t \subset s}} \omega(s) \right) \\
&\stackrel{(**)}{=} \sum_{t \in \Sigma(k, \Gamma)} \frac{(n-k)(k+2)!(k+2)^p \omega(t)}{(k+2)!(k+1)!|\Gamma_t|} \|\phi(t)\|_E^p \\
&= (n-k)(k+2)^p \|\phi\|_{(k,p)}^p,
\end{aligned}$$

where we used Proposition 2 in (*) followed by Proposition 1 in (**). \square

Similarly to [BS], we also have the following useful point-wise expression for the differential.

Proposition 32. For $\phi \in L^{(k,p^*)}(X, E^*)$ and $\tau \in \Sigma(k-1)$

$$\delta\phi(\tau) = \sum_{\substack{v \in \Sigma(0) \\ v * \tau \in \Sigma(k)}} \frac{\omega(v * \tau)}{\omega(\tau)} \phi(v * \tau).$$

Proof. Let $\psi \in L^{(k-1,p)}(X, E)$. The claim follows by a straightforward computation,

$$\begin{aligned}
\langle \phi, d\psi \rangle_k &= \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \langle \phi(\sigma), d\psi(\sigma) \rangle_E \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \langle \phi(\sigma), \sum_{i=0}^{k-1} (-1)^i \psi(\sigma_i) \rangle_E \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{\omega(\sigma)}{(k+1)!|\Gamma_\sigma|} \langle \phi(\sigma), \frac{1}{(k-1+1)!} \sum_{\substack{\tau \in \Sigma(k-1) \\ \tau \subset \sigma}} [\sigma : \tau] \psi(\tau) \rangle_E \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\tau \in \Sigma(k-1) \\ \tau \subset \sigma}} [\sigma : \tau] \frac{\omega(\sigma) \langle \phi(\sigma), \psi(\tau) \rangle_E}{(k+1)!|\Gamma_\sigma|k!} \\
&\stackrel{(*)}{=} \sum_{\tau \in \Sigma(k-1, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} [\sigma : \tau] \frac{\omega(\sigma) \langle \phi(\sigma), \psi(\tau) \rangle_E}{(k+1)!|\Gamma_\tau|k!} \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{\omega(\tau)}{\omega(\tau)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} [\sigma : \tau] \frac{\omega(\sigma) \langle \phi(\sigma), \psi(\tau) \rangle_E}{(k+1)!|\Gamma_\tau|k!} \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{\omega(\tau)}{k!|\Gamma_\tau|} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} [\sigma : \tau] \frac{\omega(\sigma) \langle \phi(\sigma), \psi(\tau) \rangle_E}{(k+1)! \omega(\tau)} \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{\omega(\tau)}{k!|\Gamma_\tau|} \sum_{\substack{v \in \Sigma(0) \\ v * \tau \in \Sigma(k)}} [v\tau : \tau] \frac{\omega(v * \tau)}{\omega(\tau)} \langle \phi(v * \tau), \psi(\tau) \rangle_E \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{\omega(\tau)}{k!|\Gamma_\tau|} \left\langle \sum_{\substack{v \in \Sigma(0) \\ v * \tau \in \Sigma(k)}} \frac{\omega(v * \tau)}{\omega(\tau)} \phi(v * \tau), \psi(\tau) \right\rangle_E = \langle \delta\phi, \psi \rangle_{k-1},
\end{aligned}$$

where we used Proposition 2 in $(*)$ above. \square

6 Localization and restriction

In this section we recall the concept of localization following [BS] and develop the notion in the setting of reflexive Banach spaces. We also consider the concept of restriction, recently considered by I. Oppenheim in the context of L^2 -cohomology [O]. Proposition 47 and 55 are the key results. The former relates the norm of the average to the norm of the differential, whereas the latter gives a global vanishing condition in the kernel of the full codifferential.

Definition 33. For a weight ω , define the localized weight as

$$\omega_\tau(\sigma) = \omega(\tau * \sigma)$$

for $\sigma \in \Sigma_\tau(j)$ and $\tau \in \Sigma(l)$ such that $\tau * \sigma \in \Sigma(j+l+1)$.

In other words, for $\tau \in \Sigma(l)$, $\omega_\tau(\sigma)$ is the number of $(n-l-1)$ -simplexes in X_τ containing $\sigma \in \Sigma_\tau(j)$.

Lemma 34. Γ_τ acts by simplicial automorphisms on X_τ .

Proof. Let $\sigma \in X_\tau$ and suppose $g \in \Gamma_\tau$. Since, $\sigma \subset \tau * \sigma$, the join of σ and τ , it follows that $g \cdot \sigma \subset g \cdot \tau * g \cdot \sigma = \tau * g \cdot \sigma$ as Γ acts by simplicial automorphisms on X and $g \in \Gamma_\tau$. Thus, $g \cdot \sigma$ is a simplex in $\tau * g \cdot \sigma$, and so $g \cdot \sigma \in X_\tau$ since it is disjoint from τ . Hence, Γ_τ act by simplicial automorphisms on X_τ . \square

Lemma 35. For $\eta \in X_\tau$, $\Gamma_{\tau\eta} = \Gamma_\tau \cap \Gamma_\eta$.

Proof. $\Gamma_{\tau\eta} = \{g \in \Gamma_\tau : g \cdot \eta = \eta\} = \Gamma_\tau \cap \Gamma_\eta$. \square

Definition 36. We denote by

- i. π_τ the restriction of π to Γ_τ , that is $\pi_\tau = \pi|_{\Gamma_\tau}$;
- ii. d_τ the restriction of d to $\mathcal{E}^{(k,p)}(X_\tau, E)$, that is $d_\tau = d|_{\mathcal{E}^{(k,p)}(X_\tau, E)}$;
- iii. δ_τ the restriction of δ to $L^{(k+1,p^*)}(X_\tau, E^*)$, that is $\delta_\tau = \delta|_{L^{(k+1,p^*)}(X_\tau, E^*)}$.

Definition 37. Let

$$\mathcal{E}^{(k,p)}(X_\tau, E) = \left\{ f : \Sigma_\tau(k) \rightarrow E : \|f\|_{(k,p)}^p < \infty \right\}$$

for $X_\tau \subset X$ denote the vector space of p -summable functions with semi-norm

$$\|f\|_{(k,p)} = \left(\sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \|f(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{(k+1)! |\Gamma_\tau \sigma|} \right)^{\frac{1}{p}}.$$

Definition 38. Let $L^{(k,p)}(X_\tau, E)$ denote the subspace

$$\left\{ f \in \mathcal{E}^{(k,p)}(X_\tau, E) : f \text{ alternating and } \forall g \in \Gamma_\tau, \forall \sigma \in \Sigma_\tau(k), f(g \cdot \sigma) = \pi_{\tau g} \cdot f(\sigma) \right\}$$

of simplicial k -cochains of X_τ twisted by π_τ .

Definition 39. For $f \in \mathcal{E}^{(k,p)}(X, E)$ and $\tau \in \Sigma(j)$ such that $k-j-1 \geq 0$, the localization of f to X_τ is the function $f_\tau \in \mathcal{E}^{(k,p)}(X_\tau, E) \in \mathcal{E}^{(k-j-1,p)}(X_\tau, E)$ defined by the localization map

$$\tau : \mathcal{E}^{(k,p)}(X, E) \longrightarrow \mathcal{E}^{(k-j-1,p)}(X_\tau, E)$$

where $\forall \sigma \in \Sigma_\tau(k-j-1)$, $f_\tau(\sigma) = f(\tau * \sigma)$. Similarly we define its dual $\mathcal{E}^{(k,p^*)}(X, E^*) \longrightarrow \mathcal{E}^{(k,p^*)}(X_\tau, E^*)$, also denoted by τ .

Definition 40. For $f \in \mathcal{E}^{(k,p)}(X, E)$ and $\tau \in \Sigma(j)$ such that $k+j+1 \leq n$, the restriction of f to X_τ is the function $f^\tau \in \mathcal{E}^{(k,p)}(X_\tau, E)$ defined by the restriction map

$$\tau : \mathcal{E}^{(k,p)}(X, E) \rightarrow \mathcal{E}^{(k,p)}(X_\tau, E)$$

where $\forall \sigma \in \Sigma_\tau(k)$, $f^\tau(\sigma) = f(\sigma)$. Similarly we define its dual $\mathcal{E}^{(k,p^*)}(X, E^*) \longrightarrow \mathcal{E}^{(k-j-1,p^*)}(X_\tau, E^*)$, also denoted by τ .

Next, we consider a number of local to global equalities that will be of use. We begin by the following useful local relation:

Proposition 41. *For $f_\tau \in L^{(k,p)}(X_\tau, E)$*

$$\|f_\tau\|_{(k,p)}^p = \frac{1}{(k+1)!|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(k)} \|f_\tau(\sigma)\|_E^p \omega(\tau * \sigma).$$

Proof.

$$\begin{aligned} \|f_\tau\|_{(k,p)}^p &= \sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \|f_\tau(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{|\Gamma_{\tau\sigma}|(k+1)!} = \sum_{\sigma \in \Sigma_\tau(k)} \frac{\|f_\tau(\sigma)\|_E^p}{|\Gamma_\tau\sigma|} \frac{\omega_\tau(\sigma)}{|\Gamma_{\tau\sigma}|(k+1)!} \\ &= \sum_{\sigma \in \Sigma_\tau(k)} \|f_\tau(\sigma)\|_E^p \frac{|\Gamma_{\tau\sigma}|}{|\Gamma_\tau|} \frac{\omega_\tau(\sigma)}{|\Gamma_{\tau\sigma}|(k+1)!} = \sum_{\sigma \in \Sigma_\tau(k)} \|f_\tau(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{|\Gamma_\tau|(k+1)!}, \\ &= \frac{1}{(k+1)!|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(k)} \|f_\tau(\sigma)\|_E^p \omega(\tau * \sigma) \end{aligned}$$

where the second equality in terms of the k -simplexes of X_τ follows from the Γ_τ invariance of the norm, and the third by the fact that $|\Gamma_{\tau\sigma}|$, the size of the Γ_τ orbit of σ , is $|\Gamma_\tau|/|\Gamma_{\tau\sigma}|$.

□

Proposition 42. *Let $f \in L^{(k,p)}(X, E)$. If $k+1 \leq n$, then*

$$(n-k)\|f\|_{(k,p)}^p = \sum_{\tau \in \Sigma(0,\Gamma)} \|f^\tau\|_{(k,p)}^p.$$

Proof. For $\xi \in \Sigma(k+1)$ such that $\tau \subset \xi$, denote by $\xi - \tau$ the k -simplex in X_τ

obtained by removing the vertex τ from ξ . Now,

$$\begin{aligned}
\sum_{\tau \in \Sigma(0, \Gamma)} \|f^\tau\|_{(k,p)}^p &= \sum_{\tau \in \Sigma(0, \Gamma)} \frac{1}{(k+1)!|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(k)} \|f(\sigma)\|_E^p \omega(\tau * \sigma) \\
&= \sum_{\tau \in \Sigma(0, \Gamma)} \frac{1}{(k+1)!|\Gamma_\tau|} \sum_{\substack{\xi \in \Sigma(k+1) \\ \tau \subset \xi}} \frac{1}{(k+2)} \|f(\xi - \tau)\|_E^p \omega(\xi) \\
&= \sum_{\tau \in \Sigma(0, \Gamma)} \sum_{\substack{\xi \in \Sigma(k+1) \\ \tau \subset \xi}} \frac{1}{(k+2)!|\Gamma_\tau|} \|f(\xi - \tau)\|_E^p \omega(\xi) \\
&\stackrel{(*)}{=} \sum_{\xi \in \Sigma(k+1, \Gamma)} \sum_{\substack{\tau \in \Sigma(0) \\ \tau \subset \xi}} \frac{1}{(k+2)!|\Gamma_\xi|} \|f(\xi - \tau)\|_E^p \omega(\xi) \\
&= \sum_{\xi \in \Sigma(k+1, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \sigma \subset \xi}} \frac{1}{(k+2)!|\Gamma_\xi|} \frac{1}{(k+1)!} \|f(\sigma)\|_E^p \omega(\xi) \\
&\stackrel{(**)}{=} \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\xi \in \Sigma(k+1) \\ \sigma \subset \xi}} \frac{1}{(k+2)!|\Gamma_\sigma|} \frac{1}{(k+1)!} \|f(\sigma)\|_E^p \omega(\xi) \\
&= \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{1}{(k+2)!|\Gamma_\sigma|} \frac{1}{(k+1)!} \|f(\sigma)\|_E^p (n-k)(k+2)! \omega(\sigma) \\
&= (n-k) \|f\|_{(k,p)}^p,
\end{aligned}$$

where the first and second equality follow by Proposition 41. noting that $f^\tau(\sigma) = f(\sigma)$, writing $\xi - \tau$ as σ and accounting for ordering. $(*)$ and $(**)$ follows by switching sums by Proposition 2 and the second last equality follows by Proposition 1. \square

Proposition 43. *Let $f \in L^{(k,p)}(X, E)$ and $0 \leq j < k$. Then,*

$$(k+1)! \|f\|_{(k,p)}^p = (k-j)! \sum_{\tau \in \Sigma(j, \Gamma)} \|f_\tau\|_{(k-j-1,p)}^p.$$

Proof.

$$\begin{aligned}
\sum_{\tau \in \Sigma(j, \Gamma)} \|f_\tau\|_{(k-j-1, p)}^p &= \sum_{\tau \in \Sigma(j, \Gamma)} \sum_{\eta \in \Sigma_\tau(k-j-1)} \frac{\omega(\tau * \eta)}{(k-j)! |\Gamma_\tau|} \|f_\tau(\eta)\|_E^p \\
&= \sum_{\tau \in \Sigma(j, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \sigma = \tau * \eta}} \frac{\omega(\sigma)}{(k-j)! |\Gamma_\tau|} \|f(\sigma)\|_E^p \\
&\stackrel{(*)}{=} \sum_{\tau \in \Sigma(j, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} \frac{(k-j)!}{(k+1)!} \frac{\omega(\sigma)}{(k-j)! |\Gamma_\tau|} \|f(\sigma)\|_E^p \\
&= \sum_{\tau \in \Sigma(j, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} \frac{\omega(\sigma)}{(k+1)! |\Gamma_\tau|} \|f(\sigma)\|_E^p \\
&\stackrel{(**)}{=} \sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\tau \in \Sigma(j) \\ \tau \subset \sigma}} \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \|f(\sigma)\|_E^p \\
&\stackrel{(***)}{=} \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{(k+1)!}{(k-j)!} \frac{\omega(\sigma)}{(k+1)! |\Gamma_\sigma|} \|f(\sigma)\|_E^p \\
&= \frac{(k+1)!}{(k-j)!} \|f\|_{(k, p)}^p,
\end{aligned}$$

where the first and second equality follow by Proposition 41 and writing $\tau * \eta$ as $\sigma \in \Sigma(k)$, respectively. On the other hand, $(*)$ follows since summing over all $\sigma \in \Sigma(k)$ such that $\tau \subset \sigma$ amounts to summing over each term in the previous sum $(k+1)!/((k+1)-(j+1))! = (k+1)!/(k-j)! \times$ times recalling that ω is symmetric and f alternating. $(**)$ follows by Proposition 2, and finally $(***)$ follows since there are $(k+1)!/(k-j)!$ terms independent of τ in the sum over all $\tau \in \Sigma(j)$ with vertices in σ . \square

Corollary 44. *Suppose $f \in L^{(k, p)}(X, E)$. If $1 < k+1 \leq n$, then*

$$\sum_{\tau \in \Sigma(0, \Gamma)} \|f_\tau\|_{(k, p)}^p = \frac{n-k}{k+1} \sum_{\tau \in \Sigma(0, \Gamma)} \|f_\tau\|_{(k-1, p)}^p.$$

Proof. Follows immediately by Proposition 42 and Proposition 43 above in the case $j = 0$,

$$\frac{k!}{(k+1)!} \sum_{\tau \in \Sigma(0, \Gamma)} \|f_\tau\|_{(k-1, p)}^p = \|f\|_{(k, p)}^p = \frac{1}{(n-k)} \sum_{\tau \in \Sigma(0, \Gamma)} \|f_\tau\|_{(k, p)}^p.$$

\square

Definition 45. *Define the localized average over a cochain ϕ by the map*

$$M : L^{(k, p)}(X_\tau, E) \rightarrow L^{(k, p)}(X_\tau, E)$$

by $\phi_\tau \mapsto M\phi_\tau = \phi_\tau^0$ where

$$\phi_\tau^0(\sigma) = \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{-1} \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \phi_\tau(\sigma).$$

Similarly, we define its dual as $\overline{M} : L^{(k,p^*)}(X_\tau, E^*) \rightarrow L^{(k,p^*)}(X_\tau, E^*)$.

Corollary 46. *The map $M : L^{(k,p)}(X_\tau, E) \rightarrow L^{(k,p)}(X_\tau, E)$ and its dual \overline{M} are bounded projections onto the space of constant maps.*

Proof. M is well-defined. Towards this end, let $\phi_\tau \in L^{(k,p)}(X_\tau, E)$. Since ω_τ is symmetric and Γ_τ -invariant, and ϕ_τ is alternating and twisted by π_τ , $M\phi_\tau$ is alternating and twisted by π_τ as a finite weighted sum of such functions. Moreover,

$$\begin{aligned}
\|M\phi_\tau\|_{(k,p)}^p &= \frac{1}{(k+1)!|\Gamma_\tau|} \sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \|\phi_\tau^0(\eta)\|_E^p \\
&= \frac{1}{(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{-p} \left\| \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \phi_\tau(\sigma) \right\|_E^p \sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \\
&= \frac{1}{(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{1-p} \left\| \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \phi_\tau(\sigma) \right\|_E^p \\
&\leq \frac{C^p}{(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{1-p} \left\| \sum_{\sigma \in \Sigma_\tau(k)} \phi_\tau(\sigma) \right\|_E^p \\
&\leq \frac{C^p |\Sigma_\tau(k)|^p}{(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{1-p} \sum_{\sigma \in \Sigma_\tau(k)} \|\phi_\tau(\sigma)\|_E^p \\
&= \frac{C^p |\Sigma_\tau(k)|^p}{(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{1-p} \sum_{\sigma \in \Sigma_\tau(k)} \frac{\omega_\tau(\sigma)}{\omega_\tau(\sigma)} \|\phi_\tau(\sigma)\|_E^p \\
&\leq \frac{C^p |\Sigma_\tau(k)|^p}{D(k+1)!|\Gamma_\tau|} \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{1-p} \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \|\phi_\tau(\sigma)\|_E^p \\
&\leq \frac{C^p |\Sigma_\tau(k)|^p}{D} |\Sigma_\tau(k)|^{1-p} D^{1-p} \|\phi_\tau\|_{(k,p)}^p \\
&= \frac{(C/D)^p}{|\Sigma_\tau(k)|} \|\phi_\tau\|_{(k,p)}^p \leq (C/D)^p \|\phi_\tau\|_{(k,p)}^p
\end{aligned}$$

where

$C = \max\{\omega_\tau(\sigma) : \sigma \in \Sigma_\tau(k)\}$ which exists as $\Sigma_\tau(k)$ contains only finitely many k -simplexes;

$D = \min\{\omega_\tau(\sigma) : \sigma \in \Sigma_\tau(k)\}$.

Hence, $M\phi_\tau \in L^{(k,p)}(X_\tau, E)$ and M is well-defined and bounded. Clearly M is

linear and

$$\begin{aligned}
M^2\phi_\tau &= M\phi_\tau^0 \\
&= \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{-1} \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \phi_\tau^0(\sigma) \\
&= \left(\sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \right)^{-1} \sum_{\sigma \in \Sigma_\tau(k)} \omega_\tau(\sigma) \left(\sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \right)^{-1} \sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \phi_\tau(\eta) \\
&= \left(\sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \right)^{-1} \sum_{\eta \in \Sigma_\tau(k)} \omega_\tau(\eta) \phi_\tau(\eta) \\
&= M\phi_\tau,
\end{aligned}$$

so M is a continuous projection onto $\{f : \Sigma_\tau(k) \rightarrow E : f = \text{constant}\} \subseteq L^{(k,p)}(X_\tau, E)$. Similarly for \overline{M} \square

Proposition 47. *Let $0 \leq j < k \leq n$, $\tau \in \Sigma(j)$ and $\phi \in L^{(k,p^*)}(X, E^*)$. Then,*

1. *if $j < k - 1$, then $\delta_\tau \phi_\tau = (-1)^{j+1}(\delta\phi)_\tau$;*
2. *if $j = k - 1$, then $(-1)^k(n - k + 1)\phi_\tau^0 = \delta\phi(\tau)$ and*

$$\|\phi_\tau^0\|_{(0,p^*)}^{p^*} = \frac{\omega(\tau)}{(n - k + 1)p^{*-1}|\Gamma_\tau|} \|\delta\phi(\tau)\|_{E^*}^{p^*}.$$

Proof. (1) As $\phi \in L^{(k,p^*)}(X, E^*)$, it follows that $\phi_\tau \in L^{(k-j-1,p^*)}(X_\tau, E^*)$ where $k - j - 1 > 0$ so $\delta_\tau \phi_\tau \in L^{(k-j-2,p^*)}(X_\tau, E^*)$ and by Proposition 32,

$$\begin{aligned}
\delta_\tau \phi_\tau(\sigma) &\stackrel{(*)}{=} \sum_{\substack{v \in \Sigma_\tau(0) \\ v*\sigma \in \Sigma_\tau(k-j-1)}} \frac{\omega_\tau(v*\sigma)}{\omega_\tau(\sigma)} \phi_\tau(v*\sigma) \\
&= \sum_{\substack{v \in \Sigma_\tau(0) \\ v*\sigma \in \Sigma_\tau(k-j-1)}} \frac{\omega(\tau*v*\sigma)}{\omega(\tau*\sigma)} \phi(\tau*v*\sigma) \\
&\stackrel{(**)}{=} \sum_{\substack{v \in \Sigma_\tau(0) \\ v*\sigma \in \Sigma_\tau(k-j-1)}} (-1)^{j+1} \frac{\omega(v*\tau*\sigma)}{\omega(\tau*\sigma)} \phi(v*\tau*\sigma) \\
&= \sum_{\substack{v \in \Sigma(0) \\ v*\tau*\sigma \in \Sigma(k)}} (-1)^{j+1} \frac{\omega(v*\tau*\sigma)}{\omega(\tau*\sigma)} \phi(v*\tau*\sigma) \\
&= (-1)^{j+1} \delta\phi(\tau*\sigma) = (-1)^{j+1} (\delta\phi)_\tau(\sigma)
\end{aligned}$$

where $(*)$ follows by Proposition 32 and $(**)$ holds since ω is symmetric, ϕ alternating and $\tau \in \Sigma(j)$. As for (2), by Proposition 32 together with the fact

that ω is symmetric and ϕ antisymmetric it follows for the codifferential that

$$\begin{aligned}\delta\phi(\tau) &= \frac{1}{\omega(\tau)} \sum_{\sigma \in \Sigma_\tau(0)} \omega(\sigma * \tau) \phi(\sigma * \tau) = \frac{1}{\omega(\tau)} \sum_{\sigma \in \Sigma_\tau(0)} (-1)^{j+1} \omega(\tau * \sigma) \phi(\tau * \sigma) \\ &= \frac{1}{\omega(\tau)} \sum_{\sigma \in \Sigma_\tau(0)} (-1)^k \omega(\tau * \sigma) \phi(\tau * \sigma),\end{aligned}$$

since $j = k - 1$. Therefore, in terms of $\phi_\tau^0 \in L^{(0,p^*)}(X_\tau, E^*)$

$$\delta\phi(\tau) = \frac{(-1)^k}{\omega(\tau)} \sum_{\sigma \in \Sigma_\tau(0)} \omega_\tau(\sigma) \phi_\tau(\sigma) = \frac{(-1)^k}{\omega(\tau)} \left(\sum_{\sigma \in \Sigma_\tau(0)} \omega_\tau(\sigma) \right) \phi_\tau^0.$$

However,

$$\frac{\sum_{\sigma \in \Sigma_\tau(0)} \omega_\tau(\sigma)}{\omega(\tau)} = \frac{\sum_{\tau * \sigma \in \Sigma(j+1)} \omega(\tau * \sigma)}{\omega(\tau)} = \frac{(n-j)(j+1)!}{(j+1)!} = (n-j) = (n-k+1)$$

by Proposition 1. The factor $(j+1)!$ in the denominator corresponds to the fact that we sum over one ordering as τ is fixed. Therefore,

$$\delta\phi(\tau) = (-1)^k (n-k+1) \phi_\tau^0,$$

and once again by Proposition 41 this gives

$$\begin{aligned}\|\phi_\tau^0\|_{(0,p^*)}^{p^*} &= \frac{1}{|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(0)} \|\phi_\tau^0(\sigma)\|_{E^*}^{p^*} \omega_\tau(\sigma) = \frac{1}{|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(0)} \frac{\|\delta\phi(\tau)\|_{E^*}^{p^*}}{(n-k+1)^{p^*}} \omega_\tau(\sigma) \\ &= \frac{1}{|\Gamma_\tau|} \frac{\|\delta\phi(\tau)\|_{E^*}^{p^*}}{(n-k+1)^{p^*}} (n-k+1) \omega(\tau) = \frac{\omega(\tau) \|\delta\phi(\tau)\|_{E^*}^{p^*}}{(n-k+1)^{p^*-1} |\Gamma_\tau|}\end{aligned}$$

□

Proposition 48. *Let $\phi \in L^{(k,p)}(X, E)$ and $\tau \in \Sigma(k-1)$. Then,*

1. *if $k = 1$,*

$$d_\tau \phi_\tau(\sigma) = -(d\phi)_\tau(\sigma) + \phi(\sigma).$$

2. *if $k > 1$,*

$$d_\tau \phi_\tau(\sigma) = (d\phi)_\tau(\sigma) + \sum_{i=0}^{k-1} (-1)^i \phi(\tau_i * \sigma).$$

Proof. Suppose $k = 1$. Then, for $\sigma = x * y \in \Sigma_\tau(1)$

$$d_\tau \phi_\tau(\sigma) = \phi_\tau(y) - \phi_\tau(x) = \phi(\tau * y) - \phi(\tau * x).$$

On the other hand, as $[\tau * x * y : x * y] = 1$

$$\begin{aligned}d\phi(\tau * x * y) &= [\tau * x * y : x * y] \phi(x * y) + [\tau * x * y : \tau * y] \phi(\tau * y) + [\tau * x * y : \tau * x] \phi(\tau * x) \\ &= \phi(x * y) - \phi(\tau * y) + \phi(\tau * x),\end{aligned}$$

gives together with the expression for $d_\tau \phi_\tau$

$$d_\tau \phi_\tau(\sigma) = -d\phi(\tau * x * y) + \phi(x * y) = -(d\phi)_\tau(\sigma) + \phi(\sigma).$$

Suppose $k > 1$ and $\tau \in \Sigma(k-1)$. Then, as previously

$$d_\tau \phi_\tau(\sigma) = \phi_\tau(y) - \phi_\tau(x),$$

and the two rightmost terms are as previously the last two terms in $d\phi(\tau * \sigma)$. \square

Proposition 49. *Let $\phi \in L^{(k,p)}(X, E)$ and $\tau \in \Sigma(0)$, then*

$$(d\phi)_\tau(\sigma) = -d_\tau \phi_\tau(\sigma) + \phi(\sigma).$$

Proof. Let $\sigma \in \Sigma(k)$. Then, similarly as in Proposition 48

$$d_\tau \phi_\tau(\sigma) = \sum_{i=0}^k (-1)^i \phi_\tau(\sigma_i) = \sum_{i=0}^k (-1)^i \phi(\tau * \sigma_i) = -(d\phi)_\tau(\sigma) + \phi(\sigma).$$

\square

Corollary 50. *Suppose $\phi \in L^{(k,p)}(X, E)$, $\tau \in \Sigma(0)$ and $k+1 \leq n$. Then, if $\phi \in \ker d$,*

$$\|d_\tau \phi_\tau\|_{(k,p)} = \|\phi^\tau\|_{(k,p)}.$$

Proof. By Proposition 49 it follows since $\phi \in \ker d$ that

$$\begin{aligned} \|d_\tau \phi_\tau\|_{(k,p)}^p &= \sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \|d_\tau \phi_\tau(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{(k+1)! |\Gamma_{\tau\sigma}|} \\ &= \sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \|\phi(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{(k+1)! |\Gamma_{\tau\sigma}|} \\ &= \|\phi^\tau\|_{(k,p)}^p. \end{aligned}$$

\square

Corollary 51. *Suppose $\phi \in L^{(k,p)}(X, E)$ and $1 < k+1 \leq n$. If $\phi \in \ker d$, then*

$$\sum_{\tau \in \Sigma(0, \Gamma)} \|d_\tau \phi_\tau\|_{(k,p)}^p = \frac{n-k}{k+1} \sum_{\tau \in \Sigma(0, \Gamma)} \|\phi_\tau\|_{(k-1,p)}^p.$$

Proof. Follows by Corollary 50 and Proposition 44. \square

Corollary 52. *Suppose $\phi \in L^{(1,p)}(X, E)$. If $\phi \in \ker d$, then*

$$-(n-1)\|\phi\|_{(1,p)}^p = \sum_{\tau \in \Sigma(0, \Gamma)} \left(\|d_\tau \phi_\tau\|_{(1,p)}^p - (n-1)\|\phi_\tau\|_{(0,p)}^p \right).$$

Proof. By a direct computation,

$$\begin{aligned} &\sum_{\tau \in \Sigma(0, \Gamma)} \|d_\tau \phi_\tau\|_{(1,p)}^p - (n-1)\|\phi_\tau\|_{(0,p)}^p \stackrel{(*)}{=} \sum_{\tau \in \Sigma(0, \Gamma)} \frac{(n-1)}{2} \|\phi_\tau\|_{(0,p)}^p - (n-1)\|\phi_\tau\|_{(0,p)}^p \\ &= -\frac{(n-1)}{2} \sum_{\tau \in \Sigma(0, \Gamma)} \|\phi_\tau\|_{(0,p)}^p \stackrel{(**)}{=} -\frac{(n-1)}{2} \frac{2!}{(1-0)!} \|\phi\|_{(1,p)}^p = -(n-1)\|\phi\|_{(1,p)}^p. \end{aligned}$$

where in $(*)$ we used Corollary 51 and in $(**)$ Proposition 43. \square

Corollary 53. Suppose $\phi \in L^{(1,p)}(X, E)$. If $\phi \in \ker d$, then

$$\sum_{\tau \in \Sigma(0, \Gamma)} \|d_\tau \phi_\tau\|_{(1,p)}^p = (n-1) \|\phi\|_{(1,p)}^p.$$

Proof. Follows directly from Corollary 52 using Proposition 43 once more. \square

Definition 54. Let $\phi \in L^{(1,p)}(X, E)$. For $\tau \in \Sigma(0)$ define a p -form on $L^{(0,p)}(X_\tau, E)$ by

$$Q_\tau(\phi_\tau) = \|d_\tau \phi_\tau\|_{(1,p)}^p - \frac{(n-1)}{2} \|\phi_\tau\|_{(0,p)}^p.$$

Similarly, we define a p^* -form on $L^{(0,p^*)}(X_\tau, E^*)$.

Corollary 55. Suppose $\phi \in L^{(1,p)}(X, E)$. If $\phi \in \ker d$ then

$$\sum_{\tau \in \Sigma(0, \Gamma)} Q_\tau(\phi_\tau) = 0.$$

Proof. Follows immediately by Corollary 53 and Proposition 43. \square

7 Poincaré inequalities on finite weighted graphs

In this section we recall some basic facts concerning Poincaré inequalities on finite weighted graphs necessary for the spectral method. For details we refer to [GN, N, NY].

Proposition 56. [N] Suppose $\dim X = 2$. Then the link X_τ of every vertex of X is a finite graph. Hence, for any $p \geq 1$ the p -Poincaré inequality

$$\begin{aligned} \sum_{\sigma \in \Sigma(0, \Gamma_\tau)} \|f_\tau(\sigma) - M f_\tau(\sigma)\|_E^p \frac{\omega_\tau(\sigma)}{|\Gamma_{\tau\sigma}|} \\ \leq \kappa_p^p \sum_{\eta \in \Sigma_\tau(1, \Gamma_\tau)} \frac{1}{2} \|f_\tau(\eta_0) - f_\tau(\eta_1)\|_E^p \frac{\omega_\tau(\eta)}{|\Gamma_{\tau\eta}|} \end{aligned}$$

holds for some $\kappa_p > 0$ and all $f: \Sigma_\tau(0) \rightarrow E$. Similarly for $f: \Sigma_\tau(0) \rightarrow E^*$. \square

The infimum of the above constants κ_p is known as the Poincaré constant of the link X_τ , and denoted by $\kappa_p(X_\tau, E)$. In terms of the norms introduced previously:

Corollary 57. Let X be two dimensional. Then, for all $f \in L^{(1,p)}(X, E)$ it holds that

$$\|f_\tau - M f_\tau\|_{(0,p)} \leq \kappa_p(X_\tau, E) \|d_\tau f_\tau\|_{(1,p)},$$

for some $\kappa_p(X_\tau, E)$. Similarly for $f \in L^{(1,p^*)}(X, E^*)$,

$$\|f_\tau - \overline{M} f_\tau\|_{(0,p^*)} \leq \kappa_{p^*}(X_\tau, E^*) \|\overline{d}_\tau f_\tau\|_{(1,p^*)},$$

for some $\kappa_{p^*}(X_\tau, E^*)$.

□

Sometimes it is useful to know how Poincaré constants change under isomorphisms. The following is immediate:

Proposition 58. *Let $T : E \rightarrow F$ be a Banach space isomorphism. If*

$$\|x\|_E \leq \|T(x)\|_F \leq C\|x\|_E,$$

then $\kappa_p(X_\tau, E) \leq C\kappa_p(X_\tau, F)$.

□

For $1 < p < \infty$ we denote by L^p the Banach space $L^p(\mu)$ of p -integrable functions on a standard Borel space (Y, \mathcal{B}) with σ -finite measure μ . As such, any separable infinite-dimensional Hilbert space \mathcal{H} is isometrically isomorphic to L^2 . In particular, we have the following relation between the Poincaré constant and spectral gap:

Proposition 59. *[N] Let λ_1 be the smallest positive eigenvalue of the graph Laplacian $\Delta_+ = (\delta d)_\tau$, defined by*

$$\Delta_+ f(v) = f(v) - \frac{1}{\omega_\tau(v)} \sum_{u \in L_v} f(u),$$

where L_v denotes the link of v in X_τ , over the space $C^{(0,p)}(X_\tau, \mathbb{R})$ of real-valued functions on the vertices. For L^2 when X is 2-dimensional, $\kappa_p(X_\tau, L^2) = \lambda_1^{-1/2}$ and more generally $\kappa_p(X_\tau, L^p) = \kappa_p(X_\tau, \mathbb{R})$. □

8 L^p -cohomology and vanishing for uniformly bounded representations

Following [BS] we introduce L^p -cohomology of X with coefficients in π as a natural extension of the L^2 -cohomology for unitary representations. In particular, if π is a unitary representation $L^2 H^k(X, \pi)$, as described below, is the cohomology of the complex of mod Γ square integrable cochains of X twisted by π . The connection to property (T) is as follows: if X is a two dimensional contractible simplicial complex and Γ acts properly discontinuously and cocompactly by automorphisms on it, then Γ has property (T) if and only if $L^2 H^1(X, \pi) = 0$ for any unitary representation [HV]. As an application we derive a spectral condition for cohomological vanishing for square integrable cochains on a two dimensional simplicial complex twisted by a uniformly bounded representation.

Definition 60. *Let*

$$L^p H^k(X, \pi) = \ker (d|_{L^{(k,p)}(X,E)}) / \text{im} (d|_{L^{(k-1,p)}(X,E)})$$

denote the L^p -cohomology groups of X with coefficients twisted by π .

As the following shows, cohomological vanishing takes place when δ is bounded from below:

Proposition 61. *The map*

$$d_{k-1}|_{L^{(k-1,p)}(X,E)} : L^{(k-1,p)}(X,E) \longrightarrow \ker d_k|_{L^{(k,p)}(X,E)}$$

is onto if its adjoint

$$\delta_k : (\ker d_k|_{L^{(k,p)}(X,E)})^* \rightarrow L^{(k-1,p^*)}(X,E^*),$$

is bounded from below, that is $\exists K > 0$ such that for all $f \in (\ker d_k|_{L^{(k,p)}(X,E)})^$*

$$\|\delta_k f\|_{(k-1,p^*)} \geq K \|f\|_{(k,p^*)}.$$

If in addition d_{k-1} is injective, d_{k-1} is onto if and only if δ_k is bounded from below.

Proof. Since $d_k \circ d_{k-1}|_{L^{(k-1,p)}(X,E)} = 0$, $\text{im } d_{k-1} \subseteq \ker d_k|_{L^{(k,p)}(X,E)}$ without further assumptions. Now, assume δ_k is bounded from below. Then δ_k is injective; towards a contradiction, suppose $f, g \in (\ker d_k|_{L^{(k,p)}(X,E)})^*$ such that $f \neq g$ and $\delta_k f = \delta_k g$. Recalling that $\|\cdot\|_{(k,p^*)}$ is a norm restricted to $L^{(k,p^*)}(X,E)$ leads to a contradiction

$$0 = \|\delta_k f - \delta_k g\|_{(k-1,p^*)} = \|\delta_k(f - g)\|_{(k-1,p^*)} \geq K \|f - g\|_{(k,p^*)} > 0.$$

Thus, δ is injective. In particular, $\ker \delta_k = \{0\}$ and since $L^{(k,p)}(X,E)$ is reflexive and $\ker d_k|_{L^{(k,p)}(X,E)}$ is closed, the latter is also reflexive and

$$\begin{aligned} \text{im } d_{k-1} &= \text{Ann}(\ker \delta_k) \\ &= \{f \in (\ker d_k|_{L^{(k,p)}(X,E)})^{**} : \langle g, f \rangle_k = 0, \forall g \in \ker \delta_k\} \\ &\cong \{f \in \ker d_k|_{L^{(k,p)}(X,E)} : \langle g, f \rangle_k = 0, \forall g \in \{0\}\} \\ &= \ker d_k|_{L^{(k,p)}(X,E)}, \end{aligned}$$

so d_{k-1} is onto $\ker d_k|_{L^{(k,p)}(X,E)}$. Next, suppose d_{k-1} is onto. Since d_{k-1} is bounded, it is bounded from below by the open mapping theorem if d_{k-1} is injective. \square

This criteria is in fact related to the Poincaré constants of the links as Proposition 63 shows. This allows us to formulate a spectral condition for cohomological vanishing.

Proposition 62. *Suppose X is a 2-dimensional locally finite simplicial complex such that for any vertex τ of X the link X_τ is connected and \mathcal{H} a separable infinite-dimensional Hilbert space. Suppose there exists a constant C such that*

- i. the map $C\mathcal{I} : E \rightarrow \mathcal{H}$ where $\mathcal{I} : E \rightarrow \mathcal{H}$ is the identity map, is a Banach space isomorphism with the property $\|x\|_E \leq \|C\mathcal{I}(x)\|_{\mathcal{H}} \leq C\|x\|_E$ for all $x \in E$. Then, for $f \in L^{(1,2)}(X,E) \cap \ker d$*

$$\kappa_2(X_\tau, \mathcal{H})^{-2} \|Mf_\tau\|_{(0,2)}^2 + Q_\tau(f_\tau) \geq \frac{1}{C^2} \left(\kappa_2(X_\tau, \mathcal{H})^{-2} - \frac{1}{C^2} \right) \|f_\tau\|_{(0,2)}^2;$$

ii. the identity map $\overline{\mathcal{I}} : E^* \rightarrow \mathcal{H}$ is a Banach space isomorphism with the property $\|x\|_{E^*} \leq \|\overline{\mathcal{I}}(x)\|_{\mathcal{H}} \leq C\|x\|_{E^*}$ for all $x \in E^*$. Then, for $f \in L^{(1,2)}(X, E^*) \cap \ker \overline{d}$

$$\kappa_2(X_\tau, \mathcal{H})^{-2} \|\overline{M}f_\tau\|_{(0,2)}^2 + Q_\tau(f_\tau) \geq \frac{1}{C^2} \left(\kappa_2(X_\tau, \mathcal{H})^{-2} - \frac{1}{C^2} \right) \|f_\tau\|_{(0,2)}^2.$$

Proof. (i). As usual, write $\|\cdot\|_{(k,2)}$ for the seminorm on $\mathcal{E}^{(k,2)}(X, E)$ and write $\|\cdot\|_{(k,2),\mathcal{H}}$ for the seminorm on $\mathcal{E}^{(k,2)}(X, \mathcal{H})$. For $f \in L^{(1,2)}(X, E) \cap \ker d$

$$Q_\tau(f_\tau) = \|d_\tau f_\tau\|_{(1,2)}^2 - \frac{1}{2} \|f_\tau\|_{(0,2)}^2 \geq \|d_\tau f_\tau\|_{(1,2),\mathcal{H}}^2 - \frac{C^2}{2} \|f_\tau\|_{(0,2),\mathcal{H}}^2$$

On the other hand, by the Poincaré inequality and the Pythagorean identity

$$\|d_\tau f_\tau\|_{(1,2),\mathcal{H}}^2 \geq \kappa_2(X_\tau, \mathcal{H})^{-2} \|f_\tau\|_{(0,2),\mathcal{H}}^2 - \kappa_2(X_\tau, \mathcal{H})^{-2} \|Mf_\tau\|_{(0,2),\mathcal{H}}^2$$

Hence,

$$Q_\tau(f_\tau) \geq \kappa_2(X_\tau, \mathcal{H})^{-2} \|f_\tau\|_{(0,2),\mathcal{H}}^2 - \kappa_2(X_\tau, \mathcal{H})^{-2} \|Mf_\tau\|_{(0,2),\mathcal{H}}^2 - \frac{C^2}{2} \|f_\tau\|_{(0,2),\mathcal{H}}^2,$$

and in terms of the $\|\cdot\|_{(k,p)}$ norm

$$\kappa_2(X_\tau, \mathcal{H})^{-2} \|Mf_\tau\|_{(0,2)}^2 + Q_\tau(f_\tau) \geq \frac{1}{C^2} \left(\kappa_2(X_\tau, \mathcal{H})^{-2} - \frac{C^2}{2} \right) \|f_\tau\|_{(0,2)}^2.$$

Similarly for (ii). □

Corollary 63. Assuming Proposition 62 holds such that $\kappa_2(X_\tau, \mathcal{H}) \leq \kappa_2(X, \mathcal{H})$ for every link X_τ of X . Then, for

$$\kappa_2(X, \mathcal{H}) < \sqrt{2}C^{-1},$$

δ and $\overline{\delta}$ are bounded from below.

Proof. By Proposition 62(i)

$$\kappa_2(X_\tau, \mathcal{H})^{-2} \|Mf_\tau\|_{(0,2)}^2 + Q_\tau(f_\tau) \geq \frac{1}{C^2} \left(\kappa_2(X_\tau, \mathcal{H})^{-2} - \frac{C^2}{2} \right) \|f_\tau\|_{(0,2)}^2.$$

Thus, summing over the representatives $\tau \in \Sigma(0, \Gamma)$ gives, applying Propositions 43, 47, and 55 to the three terms respectively, that

$$\|\delta f\|_{(0,2)}^2 \geq \left(\frac{2\kappa_2(X, \mathcal{H})}{C} \right)^2 \left(\kappa_2(X_\tau, \mathcal{H})^{-2} - \frac{C^2}{2} \right) \|f\|_{(1,2)}^2.$$

So, δ is bounded from below for $\kappa_2(X, \mathcal{H}) < \sqrt{2}C^{-1}$. Similarly for $\overline{\delta}$. □

Unfortunately, at this time we do not have a proof that the differentials would be bounded from below for the general Banach space case with a suitably large estimate for the Poincaré constants.

Theorem 64. *Let X be a locally finite 2-dimensional simplicial complex, Γ a discrete properly discontinuous group of automorphisms of X and $\pi : \Gamma \rightarrow B(\mathcal{H})$ a uniformly bounded representation of Γ on a separable infinite-dimensional Hilbert space \mathcal{H} . Suppose the link X_τ of every vertex τ of X is connected and the associated Poincaré constants satisfy*

$$C < \frac{\sqrt{2}}{\kappa_2(X_\tau, \mathcal{H})}$$

for $C = \sup_{g \in \Gamma} \|\pi_g\|$. Then, $L^2 H^1(X, \pi) = 0$.

Proof. Let E be the Banach space $(\mathcal{H}, \|\cdot\|_E)$ where $\|\cdot\|_E = \sup_{g \in \Gamma} \|\pi_g(\cdot)\|_{\mathcal{H}}$. Now, π is an isometric representation on E and have the dual diagrams:

$$\begin{array}{ccc} \ker \bar{d} & \xrightarrow{\bar{i}} & L^{(1,2)}(X, E^*) \\ \bar{d} \uparrow & \searrow \delta \circ i^* \circ \bar{i} & \downarrow i^* \\ L^{(0,2)}(X, E^*) & \xleftarrow{\delta} & (\ker d)^* \end{array}$$

$$\begin{array}{ccc} L^{(0,2)}(X, E) & \xrightarrow{d} & \ker d \\ \bar{d}^* \uparrow & \searrow \bar{d}^* \circ \bar{i}^* \circ i & \downarrow i \\ (\ker \bar{d})^* & \xleftarrow{\bar{i}^*} & L^{(1,2)}(X, E) \end{array}$$

We claim that $L^2 H^1(X, E) = 0$, that is d_0 is onto $\ker d_1$. By Proposition 61 it is enough to prove that $d_0^* = \delta_1$ is bounded from below on $(\ker d_1)^*$. By construction Proposition 62 applies with $C = \sup_{g \in \Gamma} \|\pi_g\|$, and since $\kappa_2(X_\tau, \mathcal{H}) < \sqrt{2}C^{-1}$ it follows by Corollary 63 that δ_1 is bounded from below when restricted to $\ker \bar{d}$. Hence, δ_1 is bounded from below on the image of $i^* \circ \bar{i}$. Thus, if $i^* \circ \bar{i}$ is onto $(\ker d_1)^*$, then δ_1 is bounded from below on $(\ker d_1)^*$ and $\delta_1^* = d_0$ is onto, by which the claim follows. By a similar argument \bar{d}_0^* restricted to $\ker d_1$ is bounded from below, and thus

$$\bar{d}_0^* \circ \bar{i}^* \circ i : \ker d_1 \rightarrow L^{(0,p)}(X, E)$$

is bounded from below. In particular, $\bar{i}^* \circ i$ is bounded from below and hence $(\bar{i}^* \circ i)^* = i^* \circ \bar{i}$ is onto $(\ker d_1)^*$. \square

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